

# Models with Heterogeneous Agents

## Introduction

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# Overview

- "Simple" model with heterogeneous agents
  - understanding the complexity of these models
  - role of aggregate uncertainty
  - role of incomplete markets
- Solving the Aiyagari model
  - basic numerical techniques (refresher)
- Does heterogeneity matter?

# Overview continued

- Avoiding complexity
  - heterogeneity only within the period
  - partial equilibrium
  - are two agents enough?
- Other models with heterogeneity
  - New Keynesian model
  - Multiplicity & domino effects due to tax externality
  - macro model with search frictions

# First model with heterogeneous agents

- Agents are *ex ante* the same,  
but face different idiosyncratic shocks  
⇒ agents are different *ex post*
- Incomplete markets  
⇒ heterogeneity cannot be insured away

# Individual agent

- Subject to employment shocks:
  - $e_{i,t} \in \{0, 1\}$
- Incomplete markets
  - only way to save is through holding capital
  - borrowing constraint  $k_{i,t+1} \geq 0$

# Aggregate shock

- $z_t \in \{z^b, z^g\}$
- $z_t$  affects
  - ① aggregate productivity
  - ② probability of being employed

# Laws of motion

- $z_t$  can take on two values
- $e_{i,t}$  can take on two values
- probability of being (un)employed depends on  $z_t$
- transition probabilities are such that
  - unemployment rate only depends on current  $z_t$
  - thus
    - $u_t = u^b$  if  $z_t = z^b$  &
    - $u_t = u^g$  if  $z_t = z^g$
    - with  $u^b > u^g$ .

# Firm problem

$$\begin{aligned}r_t &= \alpha z_t K_t^{\alpha-1} L_t^{1-\alpha} \\w_t &= (1 - \alpha) z_t K_t^{\alpha} L_t^{1-\alpha}\end{aligned}$$

These are identical to those of the rep. agent version



# Government

$$\tau_t w_t \bar{l} (1 - u(a_t)) = \mu w_t u(a_t)$$

$$\tau_t = \frac{\mu u(a_t)}{\bar{l}(1 - u(a_t))}$$

# Individual agent

$$\max_{\{c_{i,t}, k_{i,t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1 - \tau_t) w_t \bar{l} e_{i,t} + \mu w_t (1 - e_{i,t}) + (1 - \delta) k_{i,t}$$

$$k_{i,t+1} \geq 0$$

- this is a relatively simple problem  
**if** processes for  $r_t$  and  $w_t$  are given

# Individual agent - foc

$$\frac{1}{c_{i,t}} \geq \beta E_t \left[ \frac{1}{c_{i,t+1}} (r_{t+1} + 1 - \delta) \right]$$

$$0 = k_{i,t+1} \left( \frac{1}{c_{i,t}} - \beta E_t \left[ \frac{1}{c_{i,t+1}} (r_{t+1} + 1 - \delta) \right] \right)$$

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1 - \tau_t) w_t \bar{l} e_{i,t} + \mu w_t (1 - e_{i,t}) + (1 - \delta) k_{i,t}$$

$$k_{i,t+1} \geq 0$$

# What aggregate info do agents care about?

- current **and** future values of  $r_t$  and  $w_t$
- the period- $t$  values of  $r_t$  and  $w_t$ 
  - only depend on aggregate capital stock,  $K_t$ , &  $z_t$
  - !!! In most models, prices also depend on other characteristics of the distribution

# What aggregate info do agents care about?

- the future values, i.e.,  $r_{t+\tau}$  and  $w_{t+\tau}$  with  $\tau > 0$  depend on
  - future values of mean capital stock, i.e.  $K_{t+\tau}$ , &  $z_{t+\tau}$
- $\implies$  agents are interested in all information that forecasts  $K_t$
- $\implies$  typically this includes the complete cross-sectional distribution of employment status and capital levels  
(**even when** you only forecast futures means like you do here)

## Equilibrium - first part

- Individual policy functions that solve agent's max problem
- A wage and a rental rate given by equations above.

## Equilibrium - second part

- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.

$$f_{t+1} = Y(z_{t+1}, z_t, f_t)$$

- $f_t$  = cross-sectional distribution of beginning-of-period capital and the employment status *after* the employment status has been realized.
- $z_{t+1}$  does *not* affect the cross-sectional distribution of capital
- $z_{t+1}$  does affect the *joint* cross-sectional distribution of capital and employment status

# Transition law & timing

- $f_t$  &  $z_t \implies f_t^{\text{end-of-period}}$
- $f_t^{\text{end-of-period}}$  &  $z_{t+1} \implies f_{t+1}^{\text{beginning-of-period}} \equiv f_{t+1}$



# Transition law & timing

- Let  $g_t$  be the cross-sectional distribution of capital (so without any info on employment status)
- Why can I write

$$g_{t+1} = Y_g(z_t, g_t)?$$

# Transition law & continuum of agents

$$\begin{aligned}g_{t+1} &= Y_g(z_t, g_t) \\f_{t+1} &= Y(z_{t+1}, z_t, f_t)\end{aligned}$$

## Why are these exact equations without additional noise?

- continuum of agents  $\implies$  rely on law of large numbers to average out idiosyncratic risk
- are we allowed to do this?

# Recursive equilibrium?

## Questions

- 1 Does an equilibrium exist?
  - 1 If yes, is it unique?
  
- 2 Does a recursive equilibrium exist?
  - 1 If yes, is it unique?
  - 2 If yes, what are the state variables?

# Recursive equilibrium?

Jianjun Miao (JET, 2006): a recursive equilibrium exist for following state variables:

- usual set of state variables, namely
  - individual shock,  $e_{i,t}$
  - individual capital holdings,  $k_{i,t}$
  - aggregate productivity,  $z_t$
  - joint distribution of income and capital holdings,  $f_t$
- and *cross-sectional distribution of expected payoffs*

# Unique?

Heterogeneity  $\implies$  more reasons to expect multiplicity

- my actions depend on what I think others will do
- heterogeneity tends to go together with frictions and multiplicity more likely with frictions
  - e.g. market externalities

## Wealth-recursive (WR) equilibrium

- WR equilibrium is a recursive equilibrium with only  $e_{i,t}$ ,  $k_{i,t}$ ,  $z_t$ , and  $f_t$  as state variables.  
(Also referred to as Krusell-Smith (KS) recursive equilibrium)
- Not proven that WR equilibrium exists in model discussed here  
(at least not without making unverifiable assumptions such as equilibrium is unique for all possible initial conditions)
- Kubler & Schmedders (2002) give examples of equilibria that are *not* recursive in wealth  
(i.e., wealth distribution by itself is not sufficient)

# Wealth distribution not sufficient - Example

- Static economy  
two agents,  $i = 1, 2$ , two commodities,  $j = A, B$
- Utility:  $\ln q_A + \ln q_B$
- Endowments in state I:  $\omega_{1,A} = \omega_{2,A} = 1; \omega_{1,B} = \omega_{2,B} = 1$
- Endowments in state II:  
 $\omega_{1,A} = \omega_{2,A} = 1; \omega_{1,B} = \omega_{2,B} = 10/9$
- Normalization:  $p_A = 1$

# Wealth distribution not sufficient - Example

- State I:
  - equilibrium:  $p_B = 1; q_{1,A} = q_{2,A} = 1; q_{1,B} = q_{2,B} = 1$   
wealth of each agent: = 2
- State II:
  - equilibrium:  $p_B = 0.9; q_{1,A} = q_{2,A} = 1; q_{1,B} = q_{2,B} = 10/9$   
wealth of each agent: = 2
- Thus: same wealth levels, but different outcome



# How to proceed?

- Wealth distribution may not be sufficient!
- For numerical analysis less problematic: It typically leaves stuff out  
After obtaining solution, you should check whether the approximation is accurate or not

# How to proceed?

- For now we assume that a wealth recursive equilibrium exists (or an approximation based on it is accurate)
- This is still a tough numerical problem

# If a wealth recursive equilibrium exists

- Suppose that recursive RE for usual state space exists
  - $s_{i,t} = \{e_{i,t}, k_{i,t}, s_t\} = \{e_{i,t}, k_{i,t}, z_t, f_t\}$
- Equilibrium:
  - $c(s_{i,t})$
  - $k(s_{i,t})$
  - $r(s_t)$
  - $w(s_t)$
  - $Y(z_{t+1}, z_t, f_t)$

# Alternative representation state space

- Suppose that recursive RE for usual state space exist
  - $s_{i,t} = \{e_{i,t}, k_{i,t}, s_t\} = \{e_{i,t}, k_{i,t}, z_t, f_t\}$
- What determines current shape  $f_t$ ?
  - $z_t, z_{t-1}, f_{t-1}$  or
  - $z_t, z_{t-1}, z_{t-2}, f_{t-2}$  or
  - $z_t, z_{t-1}, z_{t-2}, z_{t-3}, f_{t-3}$  or
  - $z_t, z_{t-1}, z_{t-2}, z_{t-3}, z_{t-4}, f_{t-4}$  or
  - ...

# No aggregate uncertainty

$$s_t = \lim_{n \rightarrow \infty} \{z_t, z_{t-1}, \dots, z_{t-n}, f_{t-n}\}$$

- Why is this useful from a numerical point of view?
  - when  $z_t$  is stochastic
  - when  $z_t$  is not stochastic (case of no aggregate uncertainty)

# No aggregate uncertainty

State variables

$$\lim_{n \rightarrow \infty} \{z_t, z_{t-1}, \dots, z_{t-n}, f_{t-n}\}$$

- **If**
  - ①  $z_t = z \forall t$  and
  - ② effect of initial distribution dies out
- **then**  $s_t$  constant
  - distribution still matters!
  - but it is no longer a *time-varying* argument

# Aggregation

## Statement:

*The representative agent model is silly,  
because there is no trade in this model,  
while there is lots of trade in financial assets in reality*

# Aggregation

## Statement:

*The representative agent model is silly,  
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while there is lots of trade in financial assets in reality*

## Problem with statement:

*RA is justified by complete markets  
which relies on lots of trade*



# Complete markets & exact aggregation

- economy with ex ante identical agents
- $J$  different states
- complete markets  $\implies J$  contingent claims

# Complete markets & exact aggregation

$$\begin{aligned} \max_{c_{i,t}, b_{i,t+1}^1, \dots, b_{i,t+1}^J} & \frac{(c_{i,t})^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t \left[ v(b_{i,t+1}^1, \dots, b_{i,t+1}^J) \right] \\ \text{s.t.} & c_{i,t} + \sum_{j=1}^J q^j b_{i,t+1}^j = y_{i,t} + \sum_{j=1}^J I(j^*) b_{i,t}^j \\ & b_{i,t+1}^j > \bar{b} \quad \text{with } \bar{b} < 0 \end{aligned}$$

# Euler equations individual

$$q^j (c_{i,t})^{-\gamma} = \beta \left( c_{i,t+1}^j \right)^{-\gamma} \text{prob}(j) \quad \forall j$$

This can be written as follows:

$$c_{i,t} = \left( \frac{\beta \text{prob}(j)}{q^j} \right)^{-1/\gamma} c_{i,t+1}^j \quad \forall j$$

# Aggregation

Aggregation across individual  $i$  of

$$c_{i,t} = \left( \frac{\beta \text{prob}(j)}{q^j} \right)^{-1/\gamma} c_{i,t+1}^j \quad \forall j$$

gives

$$C_t = \left( \frac{\beta \text{prob}(j)}{q^j} \right)^{-1/\gamma} C_{t+1}^j \quad \forall j,$$

which can be rewritten as

$$q^j (C_t)^{-\gamma} = \beta (C_{t+1}^j)^{-\gamma} \text{prob}(j) \quad \forall j$$

# Use equilibrium condition

- In equilibrium:
  - aggregate consumption equals aggregate income or
  - contingent claims are in zero net supply
- Thus

$$q^j (Y_t)^{-\gamma} = \beta (Y_{t+1}^j)^{-\gamma} \text{prob}(j) \quad \forall j$$

## Back to representative agent model

- Identical FOCs come out of this RA model:

$$\begin{aligned} \max_{C_t, B_{t+1}^1, \dots, B_{t+1}^J} & \frac{(C_t)^{1-\gamma}}{1-\gamma} + \beta E_t \left[ v(B_{t+1}^1, \dots, B_{t+1}^J) \right] \\ \text{s.t.} & C_t + \sum_{j=1}^J q^j B_{t+1}^j = Y_t + \sum_{j=1}^J I(j^*) B_t^j \\ & B_{t+1}^j > \bar{b} \text{ with } \bar{b} < 0 \end{aligned}$$

# Back to model with heterogeneous agents

- ① (For now) no aggregate risk  
Aiyagari model
- ② We simplify the standard setup as follows:
  - Replace borrowing constraint by penalty function  
⇒ going short is possible but costly
  - workers have productivity instead of unemployment shocks  
 $e_{i,t}$  with  $E[e_{i,t}] = 1$

# Individual agent

$$\max_{\{c_{i,t}, k_{i,t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t}) - \frac{\zeta_1}{\zeta_0} \exp(-\zeta_0 k_{i,t}) - \zeta_2 k_{i,t}$$

s.t.

$$c_{i,t} + k_{i,t} = r_t k_{i,t-1} + w_t e_{i,t} + (1 - \delta) k_{i,t-1}$$

First-order condition

$$-\frac{1}{c_{i,t}} + \zeta_1 \exp(-\zeta_0 k_{i,t}) - \zeta_2 + E_t \left[ \frac{\beta}{c_{i,t+1}} (r_{t+1} + 1 - \delta) \right] = 0$$



# Penalty function

- advantage of  $\zeta_2$  term:
  - suppose  $\bar{k}$  and  $\bar{r}$  are steady states of rep agent model
  - if

$$\zeta_2 = \zeta_1 \exp(-\zeta_0 \bar{k})$$

then steady state of this model is same

# Equilibrium

- Unit mass of workers,  $L_t = 1$
- Competitive firm  $\implies$  agent faces competitive prices
  - $w_t = (1 - \alpha) K_t^\alpha L_t^{1-\alpha} = (1 - \alpha) K_t^\alpha$
  - $r_t = \alpha K_t^{\alpha-1} L_t^\alpha = \alpha K_t^{\alpha-1}$

- No aggregate risk so

$$K_t = K$$

- How to find the equilibrium  $K$ ?

# Algorithm

- Guess a value for  $r$
- This implies values for  $K^{\text{demand}}$  and  $w$
- Solve the individual problem with these values for  $r$  &  $w$
- Simulate economy & calculate the supply of capital,  $K^{\text{supply}}$
- If  $K^{\text{supply}} < K^{\text{demand}}$  then  $r$  too low so raise  $r$ , say

$$r^{\text{new}} = r + \lambda(K^{\text{demand}} - K^{\text{supply}})$$

- Iterate until convergence

# Algorithm

Using

$$r^{\text{new}} = r + \lambda(K^{\text{demand}} - K^{\text{supply}})$$

to solve

$$K^{\text{demand}}(r) = K^{\text{supply}}(r)$$

not very efficient

- Value of  $\lambda$  may have to be very low
- More efficient to use equation solver to solve for  $r$

# Use Dynare to solve indiv. policy rule

- Specify guess for  $r$  in mother Matlab file
- Make  $r$  parameter in \*.mod file
- In mother Matlab file write  $r$  using

```
save r_file r
```

# Use Dynare to solve indiv. policy rule

- In \*.mod file use

```
load r_file  
set_param_value('r',r)
```

instead of

```
r = 0.013;
```

# Simulate yourself using Dynare solution

- ① Use values stored by Dynare or
  - ② Replace Dynare's `disp_dr.m` with my alternative
- this saves the policy functions *exactly as shown on the screen*
    - asa matrix
    - in a Matlab data file `dynarerocks.mat`
    - under the name `decision`

# Does heterogeneity matter?

- Important to distinguish between
  - (i) theoretical results
  - (ii) their quantitative importance
- Examples
  - no aggregation in presence of incomplete markets
  - Arrow's impossibility theorem



# Does incompleteness/heterogeneity matter?

- Take model with
  - infinitely-lived agents
  - no complete markets
    - e.g. agents can only borrow/lend through a safe asset
- $\implies$  no aggregation to RA model possible
- But in many models effects small
  - why does infinitely-lived agent assumption matter?

# Does incompleteness/heterogeneity matter?

- Effects often small for
  - asset prices
  - aggregate series
    - except possibly some impact on means
- Effects much bigger for
  - individual series, e.g.  $VAR(c_{i,t}) \gg VAR(C_t)$

# Avoiding complexity

- heterogeneity only within the period
- partial equilibrium
- two agents?

# Avoiding complexity

- Lesson learned above:
  - incomplete asset markets don't do much in many environments
- This implies you should
  - either use more interesting environment
  - or use complete asset markets
- This does *NOT* imply you should eliminate heterogeneity from your models

# Only heterogeneity within period

- Household with heterogeneous members within the period:
  - members are on their own and face frictions. E.g.
    - cannot transfer funds to each other
    - cannot transfer information
- At the end of period:
  - all members bring this period's revenues to household who makes savings decision

# Partial equilibrium

Which of the following two would you prefer?

- **General equilibrium** asset pricing model that generates *unrealistic* asset prices
- **Partial equilibrium** model that uses *realistic* asset prices as exogenous processes

# Partial of general equilibrium?

What about following example

- Government sets interest rates
  - !!! Government cannot set current  $r_t$  nor  $r_{t+1}$ .
  - Suppose it sets  $E_t[r_{t+1}]$ . E.g.,

$$E_t[r_{t+1}] = (1 - \rho_r)r^* + \rho_r r_t + \varepsilon_{r,t}$$

- Government supplies capital to implement this.

# Partial of general equilibrium?

- These expenditures are financed by lump sum taxes.
- State variables are
  - $k_{i,t}$
  - $e_{i,t}$
  - $z_t$
  - $K_t$  but no higher-order moments
  - $E_t [r_{t+1}]$  or ???



# Small number of agents

Consider following endowment economy

- Type 1 agent receives  $z_{1,t}$
- Type 2 agent receives  $z_{2,t}$
- average endowment  $z_t$

$$z_t = 0.5z_{1,t} + 0.5z_{2,t}$$

- agents smooth idiosyncratic risk by trading in safe bonds

# Small number of agents

$$c_{i,t}^{-\gamma} \geq q_t \beta \mathbf{E}_t \left[ c_{i,t+1}^{-\gamma} \right]$$

$$(b_{i,t+1} - \bar{b}) \left( c_{i,t}^{-\gamma} - q_t \beta \mathbf{E}_t \left[ c_{i,t+1}^{-\gamma} \right] \right) = 0$$

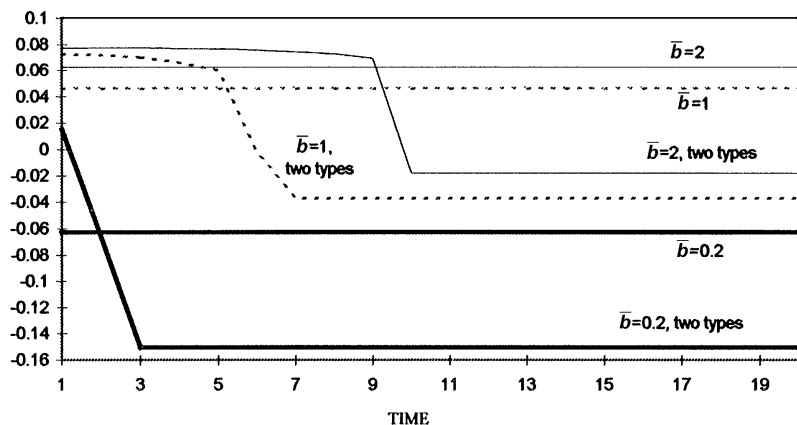
$$b_{i,t+1} \geq \bar{b}$$

$$b_{1,t+1} + b_{2,t+1} = 0$$

# Idiosyncratic risk

- You want to study effect of idiosyncratic risk.
- Suppose agent 1 repeatedly gets the bad shock
- Difference with model with lots of types?
  - here: *lots* of agents always get *same* shock at *same* time
  - so what?

# Idiosyncratic risk and interest rate



# Heterogeneity in other models

- Standard New Keynesian model
- Simple static model with tax externality
- Standard model with search friction
  - multiple steady states
  - multiple solutions

# New Keynesian models

- Calvo devil  $\implies$  heterogeneous price dispersion
- Standard approach:
  - only focus on *aggregates*
  - focus on linearized solution

# Disaggregate results in NK models

- Suppose
  - all firms start with same price (for simplicity)
  - consider monetary tightening
- Aggregate:
  - downturn because of sticky prices

# Disaggregate results in NK models

- Firms that are *not* constrained by Calvo devil:  $p_i \downarrow$ 
  - their aggregate demand  $\uparrow$  because  $p_i/P \downarrow$
  - their aggregate demand  $\downarrow$  because aggregate demand  $\downarrow$
  - total effect can easily be  $\uparrow$
- But empirical evidence suggests decline across different sectors



# Asymmetry in New Keynesian models

Suppose commodities are perfect substitutes

- Monetary tightening:
  - firms that are *not* constrained by Calvo devil:  $p_i \downarrow$
  - $\implies$  firms constrained by Calvo devil sell 0
  - $\implies$  same outcome as fully flexible case
  - $\implies \Delta Y = 0$

# Asymmetry in New Keynesian models

Suppose commodities are perfect substitutes

- Monetary stimulus:
  - Firms that are constrained by Calvo devil:  $\Delta p_i = 0$
  - $\implies$  firms *not* constrained by Calvo devil:  $\Delta p_i = 0$
  - $\implies$  same outcome as fixed  $P$  case
  - $\implies \Delta Y < 0$

# The true New Keynesian models

## **Conclusion:**

True New Keynesian models are much more interesting than the linearized version the profession is obsessed with

Is the true NK model also more realistic?

# Tax externality

- Static model
- N different skill levels
  - $z_k, k = 1, \dots, N$
  - $z_1 = \bar{z}$
  - $z_{k+1} = z_k + \varepsilon$
- unemployed get benefits

# Tax externality

*animated picture*

# Search model

Consider the following model

- unit mass of workers
- workers need to search to find a job
- employers post vacancy to find worker
- productivity of matched pairs distributed i.i.d.
  - so each period a new draw

# Key decision

- Given value of  $\varepsilon_{i,t}$  is it better to
  - ① produce or
  - ② quit and enjoy leisure?

# Equation for cut-off value

- The cut-off value  $\bar{\varepsilon}_i$  given by

$$0 = \bar{\varepsilon}_i + G - b - W$$

- $G$  : continuation value of ending period in match
  - does not depend on  $\varepsilon_{i,t}$  (i.i.d. assumption)
  - does depend on  $\bar{\varepsilon}_i$
- $W$  : continuation value of ending period not in match
  - also depends on  $\bar{\varepsilon}_i$



## Solution for cut-off value

- We are looking for a solution to

$$0 = \bar{\epsilon}_i + G(\bar{\epsilon}_i) - b - W(\bar{\epsilon}_i)$$

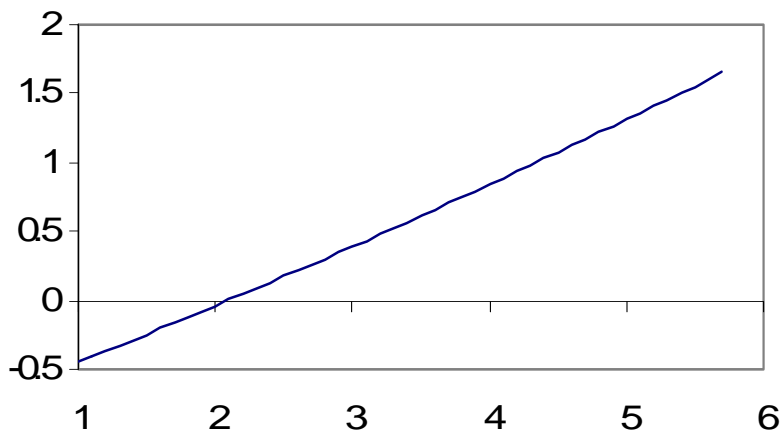
- Unique solution if

$$\frac{\partial (\bar{\epsilon}_i + G(\bar{\epsilon}_i) - b - W(\bar{\epsilon}_i))}{\partial \bar{\epsilon}_i} > 0 \quad \forall \bar{\epsilon}_i$$

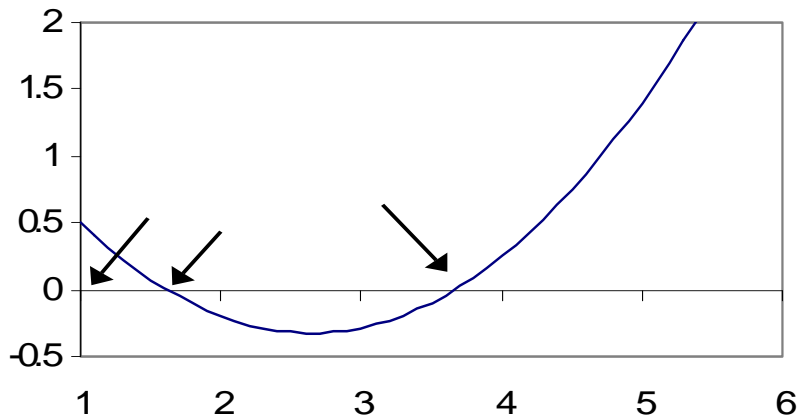
- but typically we have

$$\frac{\partial (G(\bar{\epsilon}_i) - W(\bar{\epsilon}_i))}{\partial \bar{\epsilon}_i} < 0 \quad \text{for some } \bar{\epsilon}_i$$

## Unique steady state



## Multiple steady state case



# Reasons for multiplicity

- *expectations* about the stability of future matches
  - as in example above
- market activity could affect revenues
  - as in static example with tax externality

# Tax externality and multiplicity

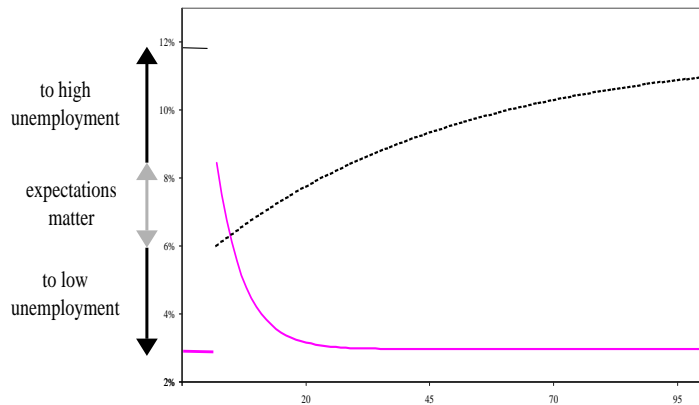
Easy to get two steady states

- Low (high) taxes  $\implies$
- Surplus high (low)  $\implies$
- Job destruction low (high)  $\implies$
- Unemployment rate low (high)  $\implies$
- Taxes indeed low (high)

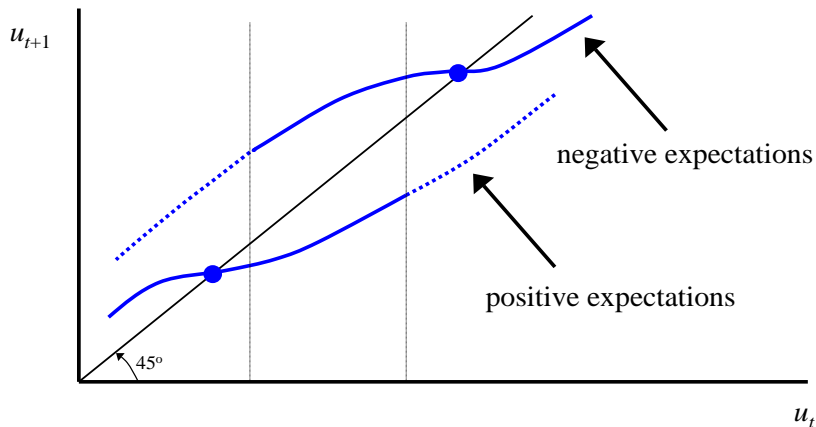
# Multiple what?

Multiple steady states  $\Rightarrow$  multiple solutions

# Transition dynamics I



# Transition dynamics II





# Why is it hard to get this published in AER?

- What aspect of distribution determines whether this is quantitatively important?
- How do you get data on this?

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